Convenzioni di segno:

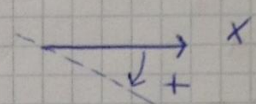
- x : Positivo da sinistra verso destra
- y : Positivo dall'alto verso il basso
- θ : Positivo orario e misurato a partire dall'asse x
- M : Positivo quando comprime le fibre superiori
- $1/\rho$: Positiva quando la concavità è verso il basso

$$\frac{d^2 v}{dx^2} = -\frac{M(x)}{EJ}$$

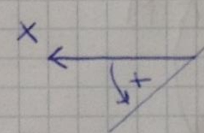
Se si inverte il segno del Momento OPPURE y allora l'equazione si scrive con il segno + -
Se si cambiano entrambi i segni di Momento e y allora l'equazione rimane così com'è!

Come spiegare i segni:

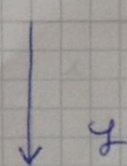
- Se x va da sinistra a destra l'angolo è preso positivo orario

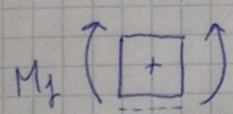


- Se x va da destra a sinistra l'angolo è preso positivo antiorario

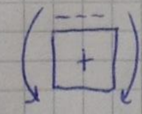


- Se y va verso il basso, allora in base a come scelgo il segno di M_f:





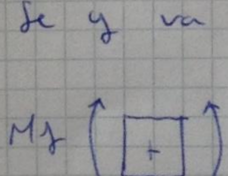
$y'' = -\frac{M_f}{EJ}$



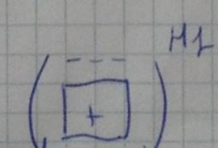
$y'' = \frac{M_f}{EJ}$

La scelta fatta del segno meno nell'equazione della linea elastica si spiega osservando che con le convenzioni adottate un tronco di trave soggetto a M_f positivo si atteggia secondo una linea che ha la concavità volta verso la direzione negativa di y e dunque derivata seconda e M_f hanno segni opposti

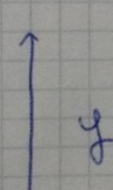
- Se y va verso l'alto

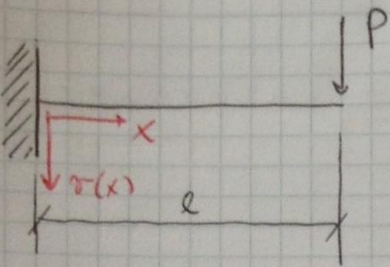


$y'' = \frac{M_f}{EJ}$

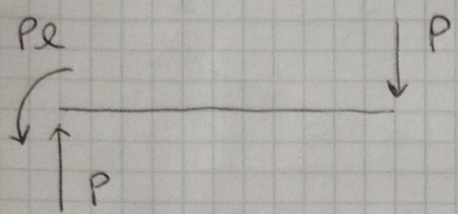


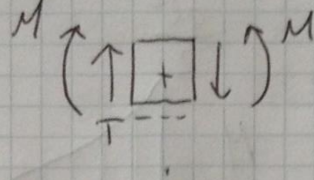
$y'' = -\frac{M_f}{EJ}$

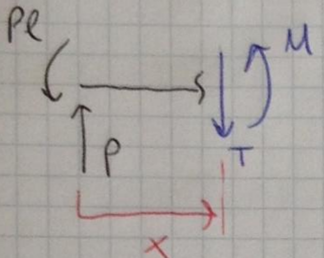




- Reazioni vincolari
- Azioni interne M e T
- Deformata
- Freccia massima

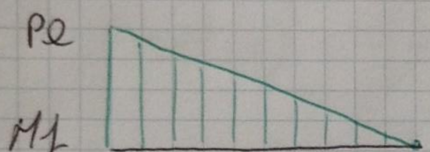


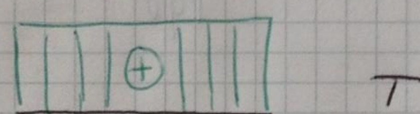


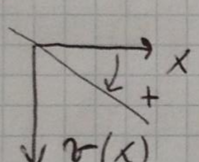


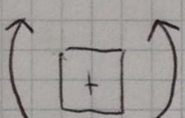
$$T = P$$

$$M + Pl - Px = 0 \qquad M = Px - Pl$$









$$v''(x) = -\frac{M(x)}{EJ}$$

$$EJ\,v'' = -M = Pl - Px$$

$$EJ\,v' = Plx - \frac{P}{2}x^2 + A$$

$$EJ\,v = \frac{Pl}{2}x^2 - \frac{P}{6}x^3 + Ax + B$$

$$v(x=0) = 0 \longrightarrow B = 0$$

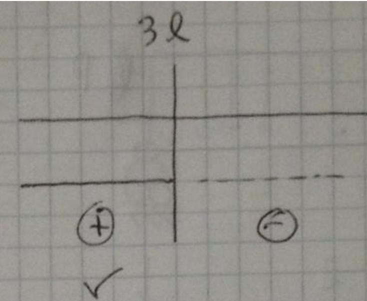
$$v'(x=0) = 0 \longrightarrow A = 0$$

$$v(x) = \frac{1}{EJ}\left(\frac{Pl}{2}x^2 - \frac{P}{6}x^3\right) = \frac{1}{EJ}x^2\left(\frac{Pl}{2} - \frac{P}{6}x\right)$$

$$v'(x) = \frac{1}{EJ}\left(Plx - \frac{P}{2}x^2\right) = \frac{1}{EJ}x\left(Pl - \frac{P}{2}x\right)$$

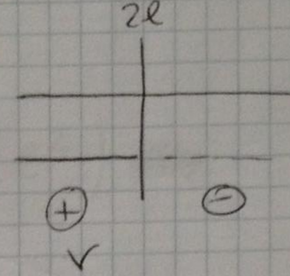
$v(x) > 0$ $\quad x^2 > 0$

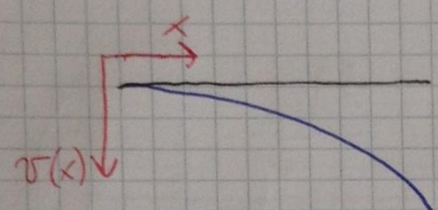
$\frac{Pl}{\cancel{2}} - \frac{P}{\cancel{6}_3}x > 0 \quad x < 3l$



$v'(x) > 0$ $\quad x > 0$

$Pl - \frac{P}{2}x > 0 \quad x < 2l$



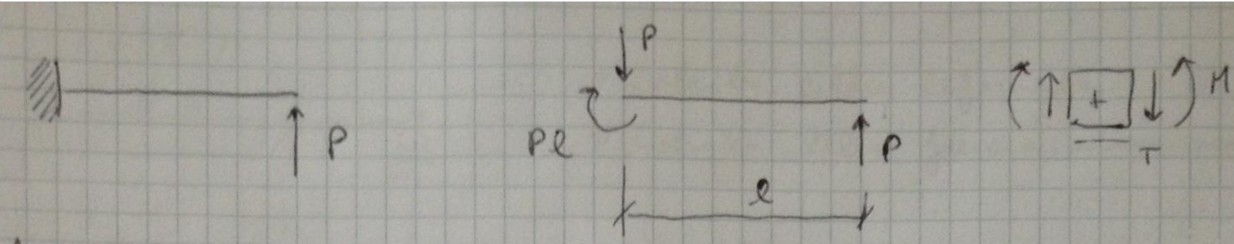


$v''(x) > 0$

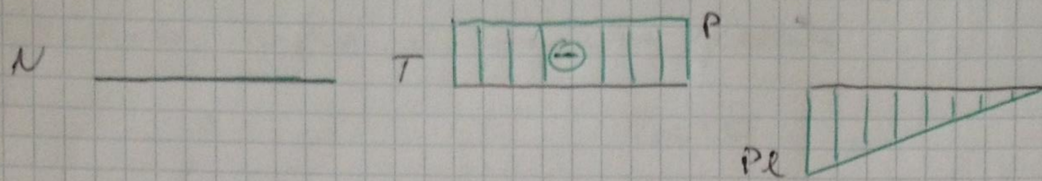
$Pl - Px > 0 \quad x < l$

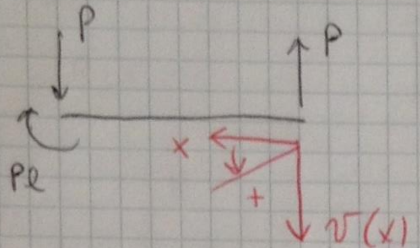
Freccia massima:

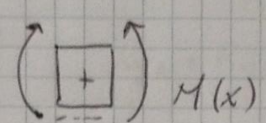
$$v_{max}(x = l) = \frac{1}{EJ}\left(\frac{Pl}{2}l^2 - \frac{Pl^3}{6}\right) = \frac{Pl^3}{3EJ}$$



Azioni interne







$$v''(x) = -\frac{M(x)}{EJ}$$

$EJ\, v'' =$ [redacted] $- Px$

Condizioni al contorno:

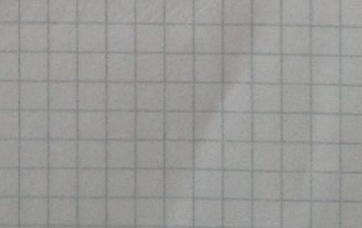
$EJ\, v' = -\frac{1}{2}Px^2 + A$ $\qquad v'(l) = 0$

$EJ\, v = -\frac{1}{6}Px^3 + Ax + B$ $\qquad v(l) = 0$

$$v'(x=l) = \frac{1}{EJ}\left(-\frac{1}{2}Pl^2 + A\right) = 0 \qquad A = \frac{Pl^2}{2}$$

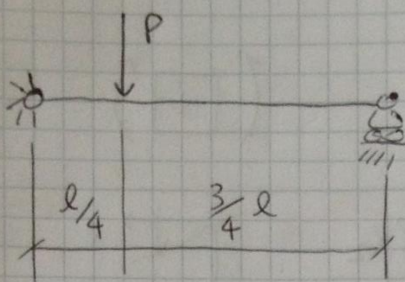
$$v(x=l) = \frac{1}{EJ}\left(-\frac{Pl^3}{6} + \frac{Pl^3}{2} + B\right) = 0 \qquad B = -\frac{Pl^3}{3}$$

$$v(x) = -\frac{Px^3}{6} + \frac{Pl^2}{2}x - \frac{Pl^3}{3}$$



$v'(x) = -\frac{Px^2}{2} + \frac{Pl^2}{2}$ [redacted] > 0

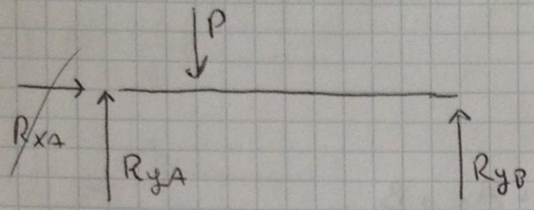
[redacted] $x^2 < l^2 \qquad -l < x < l$



- Calcolo della deformata
- Calcolo della freccia massima

1) → Reazioni vincolari

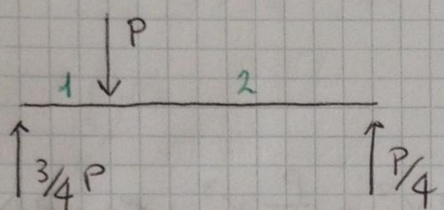
2) → Calcolo taglio e momento flettente

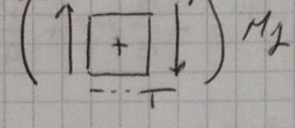


$R_{xA} = 0$

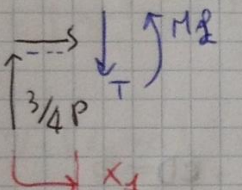
$\sum M_A = 0 \quad P\frac{l}{4} - R_{yB}\, l = 0 \quad R_{yB} = \frac{P}{4}$

$R_{yA} + R_{yB} - P = 0 \quad R_{yA} = P - R_{yB} = \frac{3}{4}P$





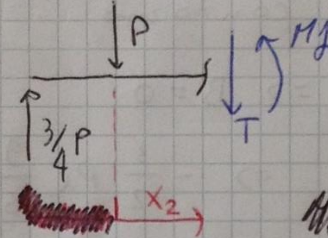
Tratto 1



$T = \frac{3}{4}P$

$M_f = \frac{3}{4}P x_1$

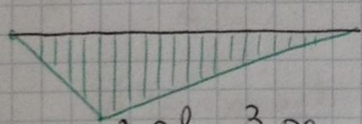
Tratto 2

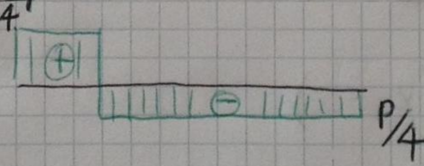


$T = \frac{3}{4}P - P = -\frac{P}{4}$

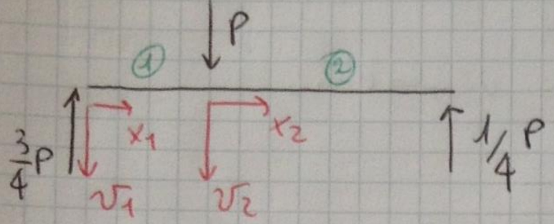
$M_f + P x_2 - \frac{3}{4}P\left(x_2 + \frac{l}{4}\right) = 0$

$M_f = -P x_2 + \frac{3}{4}P x_2 + \frac{3}{16} l P$



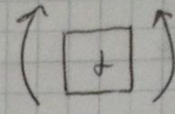


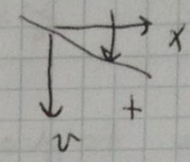
Linea elastica:



$$M_{f1} = \frac{3}{4} P x_1$$

$$M_{f2} = \frac{3}{16} Pl - \frac{1}{4} P x_2$$





$$v_1'' = -\frac{M_1}{EJ} \qquad EJ v_1'' = -\frac{3}{4} P x_1$$

$$EJ v_1' = -\frac{3}{4}\frac{1}{2} P x_1^2 + A \qquad EJ v_1 = -\frac{3}{24} P x_1^3 + Ax + B$$

$$v_2'' = -\frac{M_2}{EJ} \qquad EJ v_2'' = -\frac{3}{16} Pl + \frac{P}{4} x_2$$

$$EJ v_2' = -\frac{3}{16} Pl x_2 + \frac{P}{8} x_2^2 + C \qquad EJ v_2 = -\frac{3}{32} Pl x_2^2 + \frac{P}{24} x_2^3 + C x_2 + D$$

Condizioni al contorno

$$v_1(x_1 = 0) = 0 \implies B = 0$$

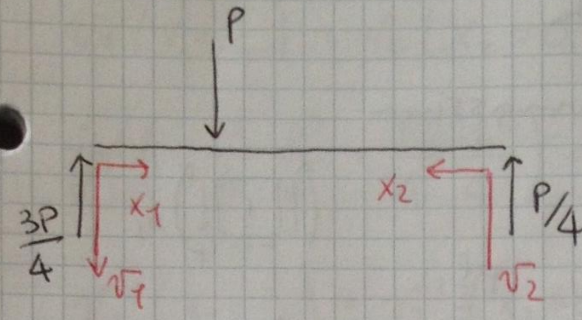
$$v_2\left(x_2 = \frac{3}{4} l\right) = 0 \implies -\frac{3}{32} Pl \frac{9}{16} l^2 + \frac{P}{24}\frac{27}{64} l^3 + C\frac{3}{4} l + D = 0$$

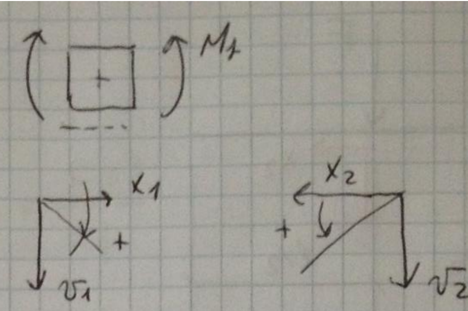
$$v_1\left(x_1 = \frac{l}{4}\right) = v_2\left(x_2 = 0\right)$$

$$-\frac{3}{24} P \frac{l^3}{64} + A\frac{l}{4} = D$$

$$v'\left(x_1 = \frac{l}{4}\right) = v_2'\left(x_2 = 0\right)$$

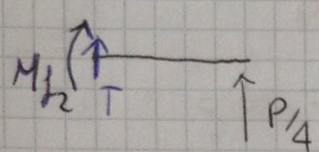
$$-\frac{3}{8} P \frac{l^2}{16} + A = C$$





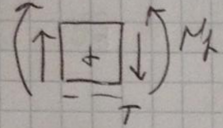
1) $M_{f1} = \frac{3}{4} P x_1$

2)



$T = -P/4$

$M_{f2} = \frac{P}{4} x_2$



$v_2'' = -\frac{M_{f2}}{EJ}$ $\qquad EJ\, v_2'' = -\frac{P}{4} x_2$

$EJ\, v_2' = -\frac{P}{8} x_2^2 + C \qquad EJ\, v_2 = -\frac{P}{24} x_2^3 + C x_2 + D$

$v_1(x_1 = 0) = 0 \Rightarrow B = 0$

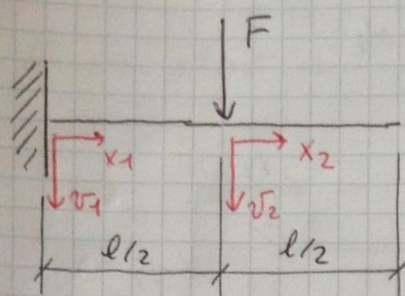
$v_2(x_2 = 0) = 0 \Rightarrow D = 0$

$v_1\left(x_1 = \frac{l}{4}\right) = v_2\left(x_2 = \frac{3}{4} l\right)$

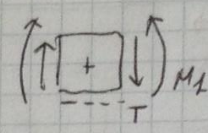
$v_1'\left(x_1 = \frac{l}{4}\right) = -v_2'\left(x_2 = \frac{3}{4} l\right)$

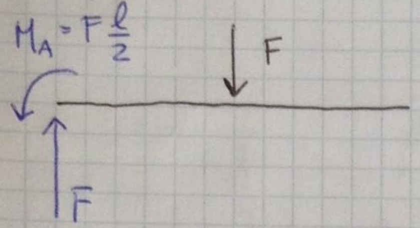
Attenzione!

Due equazioni in due incognite (A e C)

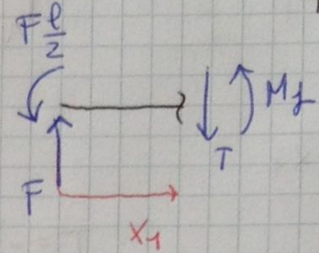


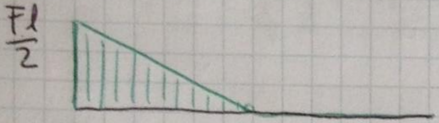
- Deformata
- Freccia massima





$M_A = F\frac{l}{2}$

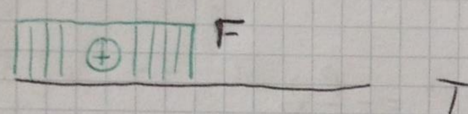




$Mf + F\frac{l}{2} - Fx = 0$

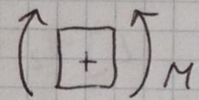
$Mf = Fx - F\frac{l}{2}$

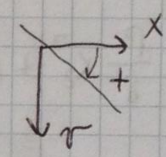
$T = F$



Tratto 1: $Mf = Fx - F\frac{l}{2}$

Tratto 2: $Mf = 0$





Tratto 1: $v'' = -\frac{M}{EJ}$ $\quad EJv'' = -Fx + F\frac{l}{2}$

$EJv' = -\frac{F}{2}x^2 + \frac{Fl}{2}x + A \quad EJv = -\frac{F}{6}x^3 + \frac{Fl}{4}x^2 + Ax + B$

Posso imporre subito le 2 condizioni al contorno

$v_1(x=0) = 0 \rightarrow B = 0$

$v_1'(x=0) = 0 \rightarrow A = 0$

$v_1(x_1) = \frac{1}{EJ}\left(-\frac{F}{6}x^3 + \frac{Fl}{4}x^2\right)$

Tratto 2

$EJv_2'' = -\frac{M}{EJ} = 0 \quad EJv_2' = C \quad EJv_2 = Cx + D$

$$v_1\left(x_1=\frac{l}{2}\right)=v_2\left(x_2=0\right)$$

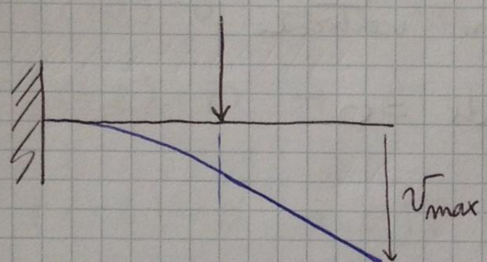
$$\frac{1}{\cancel{EJ}}\left(-\frac{F}{6}\frac{l^3}{8}+\frac{Fl}{4}\frac{l^2}{4}\right)=\frac{1}{\cancel{EJ}}D \qquad D=\frac{Fl^3}{24}$$

$$v_1'\left(x_1=\frac{l}{2}\right)=v_2'\left(x_2=0\right)$$

$$\frac{1}{\cancel{EJ}}\left(-\frac{F}{2}\frac{l^2}{4}+\frac{Fl}{2}\frac{l}{2}\right)=\frac{1}{\cancel{EJ}}C \qquad C=\frac{Fl^2}{8}$$

$$v_2'=\frac{Fl^2}{8}$$

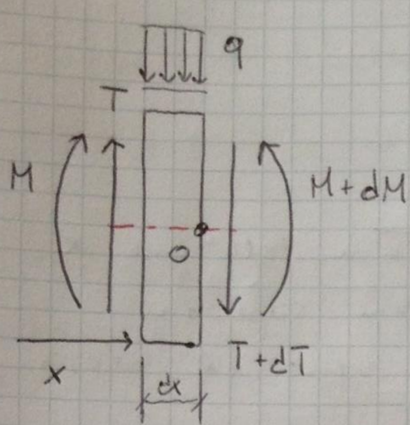
$$v_2=\frac{Fl^2}{8}x_2+\frac{Fl^3}{24}$$

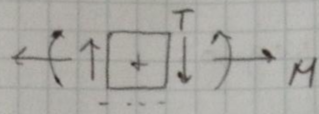


$$v_{max}=v_2\left(x_2=\frac{l}{2}\right)=$$

$$=\left(\frac{Fl^2}{8}\frac{l}{2}+\frac{Fl^3}{24}\right)\frac{1}{EJ}=$$

$$=\frac{5Fl^3}{48EJ}$$



Convenzione 

Equilibrio rispetto a O:

$$T\,dx + \cancel{M} - \cancel{M} - dM - \cancel{q\frac{dx^2}{2}} = 0$$

~ 0 (trascuro)

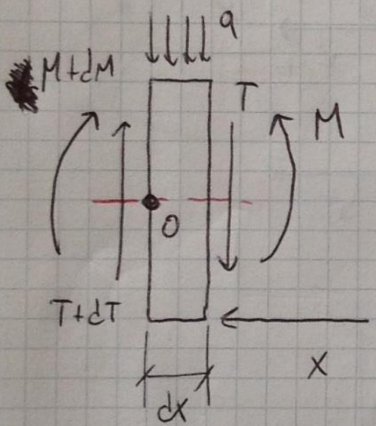
$$T = \frac{dM}{dx}$$

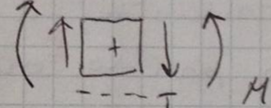
il segno di questa relazione dipende dalla convenzione adottata.

Equilibrio verticale:

$$\cancel{T} - \cancel{T} - dT - q\,dx = 0 \qquad \frac{dT}{dx} = -q$$

Attenzione alla direzione di x che se, pur mantenendo le stesse convenzioni sul taglio, viene invertita, anche il segno della derivata del taglio si inverte.



Convenzione 

Equilibrio rispetto a O:

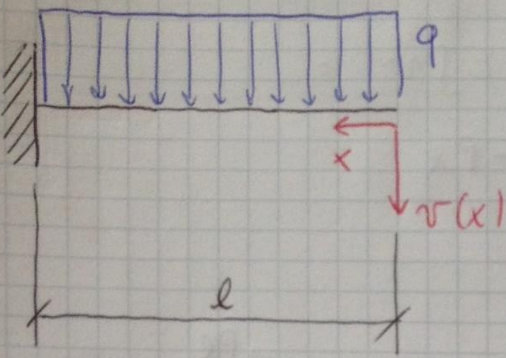
$$\cancel{M} + dM - \cancel{M} + \cancel{q\,dx\frac{dx}{2}} + T\,dx = 0$$

~ 0

$$T = -\frac{dM}{dx}$$

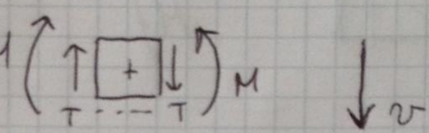
Equilibrio verticale:

$$\cancel{T} + dT - \cancel{T} - q\,dx = 0 \qquad \frac{dT}{dx} = q$$



- Deformata qualitativa
- Diagramma delle azioni interne di M e T senza calcolo reazioni vincolari.
- Calcolo reazioni vincolari a partire dai diagrammi M e T

Convenzione linea elastica



$$\Rightarrow \quad v''(x) = -\frac{M(x)}{EJ} \qquad\qquad T = -\frac{dM}{dx}$$

$$v'''(x) = -\frac{dM}{dx\,EJ} = \frac{T}{EJ} \qquad\qquad \frac{dT}{dx} = q$$

$$v^{IV}(x) = \frac{dT}{dx}\frac{1}{EJ} = \frac{q}{EJ}$$

$$EJ\,v^{IV}(x) = q$$

$$EJ\,v'''(x) = qx + \cancel{A}$$

$$EJ\,v''(x) = \frac{q}{2}x^2 + \cancel{Ax} + \cancel{B}$$

$$EJ\,v'(x) = \frac{q}{6}x^3 + \cancel{\frac{A}{2}x^2} + \cancel{Bx} + C$$

$$EJ\,v(x) = \frac{q}{24}x^4 + \cancel{\frac{A}{6}x^3} + \cancel{\frac{B}{2}x^2} + Cx + D$$

Condizioni al contorno: (Partiamo da quelle "furbe")

$$T(x=0) = 0 \longrightarrow T = EJ\,v'''(x) = qx + A \xrightarrow{x=0} A = 0$$

$$M(x=0) = 0 \longrightarrow M = -EJ\,v''(x) = -\frac{q}{2}x^2 - Ax - B = 0 \xrightarrow{x=0} B = 0$$

$$v'(x=l) = 0$$

$$EJ\,v'(x=l) = \frac{q}{6}l^3 + C = 0 \rightarrow C = -\frac{q}{6}l^3$$

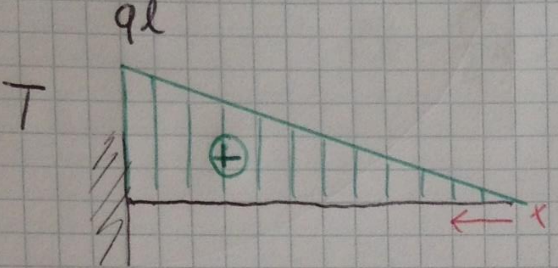
$$v(x=l) = 0$$

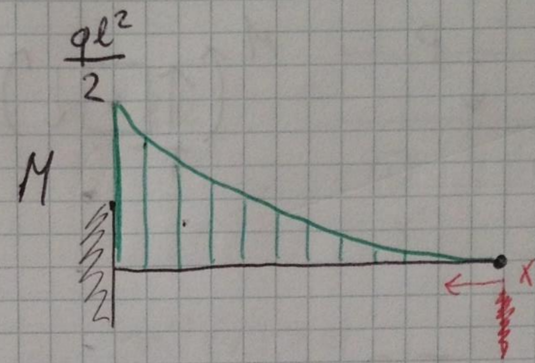
$$EJ\,v(x=l) = \frac{q}{24}l^4 - \frac{q}{6}l^3\,l + D = 0 \longrightarrow D = +\frac{1}{8}ql^4$$

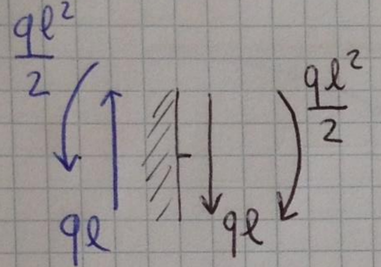
$$M = -\frac{q}{2}x^2$$

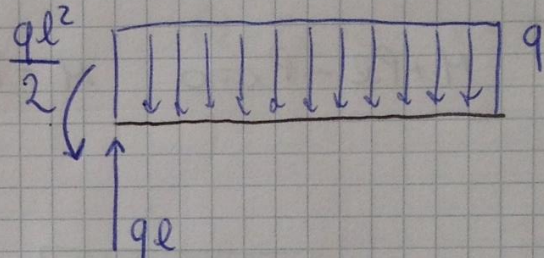
$$T = qx$$

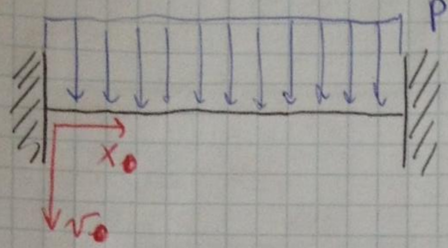
$$v(x) = \frac{1}{EJ}\left(\frac{q}{24}x^4 - \frac{q}{6}\ell^3 x + \frac{q\ell^4}{8}\right)$$







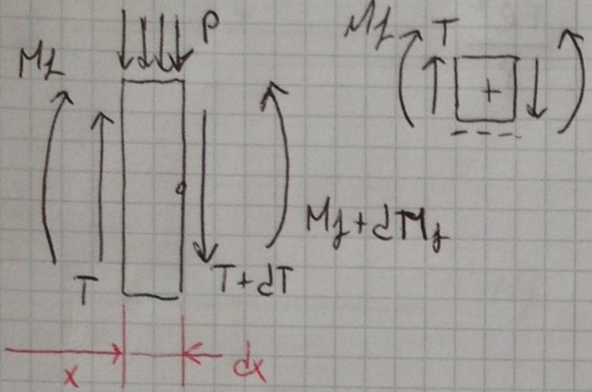




Struttura 3 volte iperstatica

- Deformata
- Diagrammi M e T (azioni interne)

Dall'equilibrio su una "fetta" di sezione:



Equilibrio dei momenti rispetto ad O:

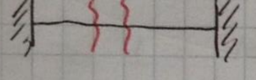
$$T\,dx + \cancel{M_f} - (\cancel{M_f} + dM_f) - P\,\cancel{dx}\,\frac{dx}{2} = 0$$

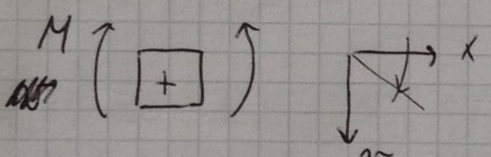
$$T = \frac{dM_f}{dx}$$

Equilibrio forze verticali

$$\cancel{T} - (\cancel{T} + dT) - P\,dx = 0$$

$$\frac{dT}{dx} = -P$$





Convenzioni linea elastica

$$v'' = -\frac{M}{EJ} \longrightarrow \frac{dM}{dx} = T$$

$$v''' = -\frac{dM}{dx}\frac{1}{EJ} = -\frac{T}{EJ} \qquad v^{IV} = -\frac{dT}{dx\,EJ} = \frac{P}{EJ}$$

$$v^{IV} = \frac{P}{EJ}$$ vale nel tratto di trave interessata dal carico distribuito.

Integriamo!

$$v^{IV} = \frac{P}{EJ} \longrightarrow EJ\,v^{IV} = P$$

$$EJ\,v''' = Px + A$$

$$EJ\,v'' = P\frac{1}{2}x^2 + Ax + B$$

$$EJ\,v' = \frac{1}{6}Px^3 + \frac{A}{2}x^2 + Bx + C$$

$$EJ\,v = \frac{P}{24}x^4 + \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D$$

Condizioni al contorno:

$$v(x=0) = 0 \rightarrow D = 0$$

$$v'(x=0) = 0 \rightarrow C = 0$$

$$v(x=l) = 0 \rightarrow EJ\,v = \frac{Pl^{\cancel{4}\,2}}{24} + \frac{A}{6}l^{\cancel{3}} + \frac{Bl^{\cancel{2}}}{2} = 0$$

$$v'(x=l) = 0 \rightarrow EJ\,v' = \frac{Pl^{\cancel{3}\,2}}{6} + \frac{A}{2}l^{\cancel{2}} + B\cancel{l} = 0$$

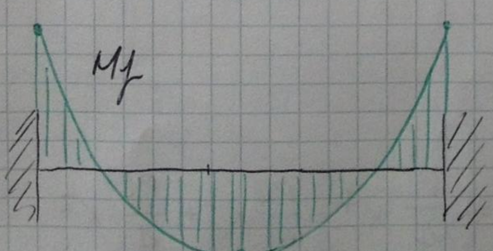
$$B = -\frac{Pl^2}{6} - \frac{Al}{2}$$

$$\frac{Pl^{\cancel{2}}}{24} + \frac{A}{6}\cancel{l} - \frac{Pl^{\cancel{2}}}{12} - \frac{A\cancel{l}}{4} = 0 \qquad \frac{A}{12} = -\frac{Pl}{24} \qquad A = -\frac{Pl}{2}$$

$$B = -\frac{Pl^2}{6} + \frac{Pl^2}{4} = \frac{Pl^2}{12}$$

$$\cancel{EJ}\,v''(x) = \left(\frac{P}{2}x^2 - \frac{Pl}{2}x + \frac{Pl^2}{12}\right)\frac{1}{EJ} = -\frac{M}{EJ}$$

$$-M(x) = \frac{P}{2}x^2 - \frac{Pl}{2}x + \frac{Pl^2}{12} \qquad M(x) = -\frac{P}{2}x^2 + \frac{Pl}{2}x - \frac{Pl^2}{12}$$

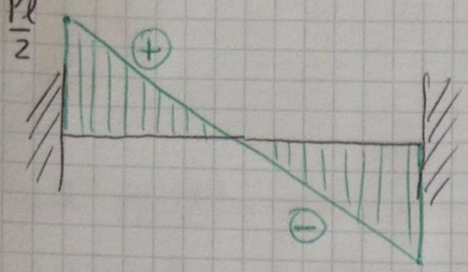


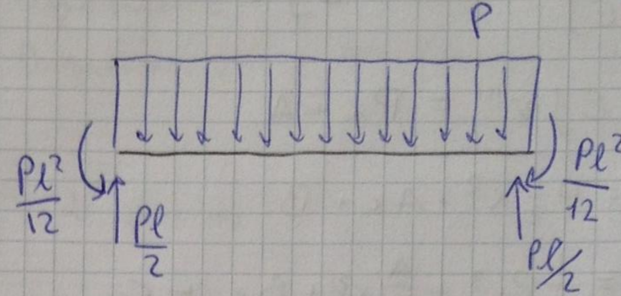
$$x = 0 \qquad M = -\frac{Pl^2}{12}$$

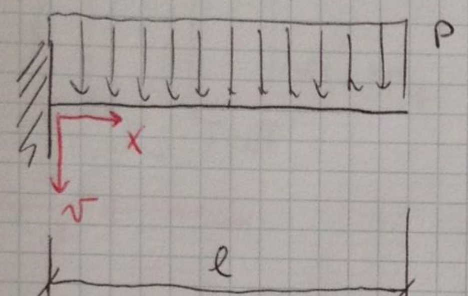
$$x = \frac{l}{2} \qquad M = \frac{Pl^2}{24}$$

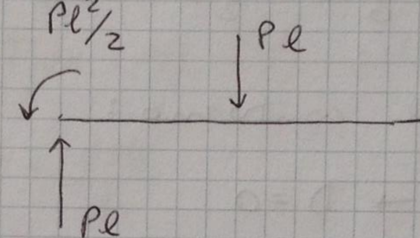
$$x = l \qquad M = -\frac{Pl^2}{12}$$

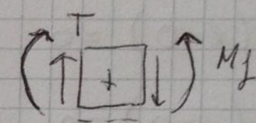
$$T = \frac{dM}{dx} = -Px + \frac{Pl}{2}$$

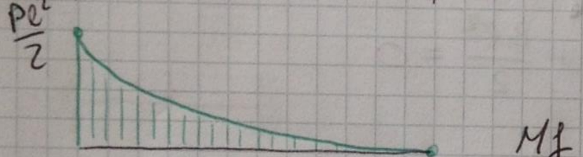


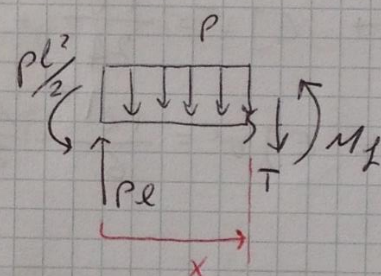


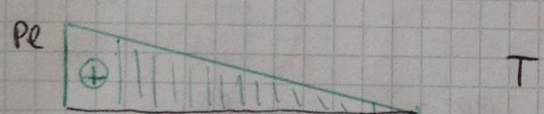








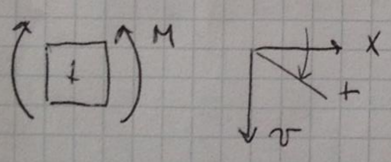




$$T + Px - Pl = 0 \qquad T = Pl - Px$$

$$Mf + P\frac{x^2}{2} + P\frac{l^2}{2} - Plx = 0$$

$$Mf = -P\frac{x^2}{2} + Plx - P\frac{l^2}{2}$$



$$v'' = -\frac{M}{EJ} \longrightarrow EJv'' = P\frac{x^2}{2} - Plx + P\frac{l^2}{2}$$

$$EJv' = P\frac{1}{6}x^3 - \frac{Pl}{2}x^2 + P\frac{l^2}{2}x + A$$

$$EJv = \frac{P}{24}x^4 - \frac{Pl}{6}x^3 + \frac{Pl^2}{4}x^2 + Ax + B$$

$$v(x=0) = 0 \longrightarrow B = 0$$

$$v'(x=0) = 0 \longrightarrow A = 0$$

$$EJv = \frac{P}{24}x^4 - \frac{Pl}{6}x^3 + \frac{Pl^2}{4}x^2$$

Si poteva risolvere anche utilizzando

$$v^{IV} = \frac{P}{EJ}$$

Vediamo come:

$$v^{IV} = \frac{P}{EJ} \qquad EJ\, v^{IV} = P$$

$$EJ\, v''' = Px + A$$

$$EJ\, v'' = P\tfrac{1}{2}x^2 + Ax + B$$

$$EJ\, v' = \frac{P}{6}x^3 + \frac{A}{2}x^2 + Bx + C$$

$$EJ\, v = \frac{P}{24}x^4 + \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D$$

$v(x=0) = 0$ e $v'(x=0) = 0 \Rightarrow D = C = 0$

Due condizioni aggiuntive

$M(l) = 0$
$T(l) = 0$

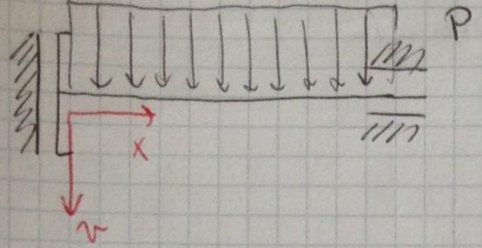
$$v''(l) = -\frac{M}{EJ} = 0 \rightarrow \frac{P}{2}l^2 + Al + B = 0$$

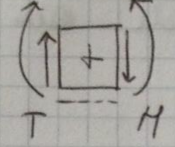
$$\frac{dM}{dx} = T$$

$$EJ\, v'' = -M \rightarrow M = -P/_2\, x^2 - Ax + B$$

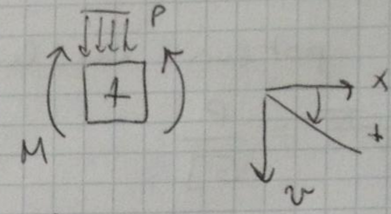
$$\frac{dM}{dx} = -Px - A \Rightarrow T\Big|_{(l)} = 0 = -A = 0 \qquad A = 0$$

$$B = -\frac{Pl^2}{2}$$





Segni per azioni interne



Segni per linea elastica

$$v^{IV} = \frac{P}{EJ}$$

$$EJ\,v^{IV} = P$$

$$EJ\,v''' = Px + \cancel{A}$$

$$EJ\,v'' = \frac{P}{2}x^2 + \cancel{Ax} + B$$

$$EJ\,v' = \frac{P}{6}x^3 + \cancel{\frac{A}{2}x^2} + Bx + \cancel{C}$$

$$EJ\,v = \frac{P}{24}x^4 + \cancel{\frac{A}{6}x^3} + \frac{B}{2}x^2 + \cancel{Cx} + D$$

Condizioni al contorno:

$$v'(x=0) = 0 \longrightarrow C = 0$$

$$T(x=0) = 0 \quad (\text{pattino}) \longrightarrow -P = \frac{dT}{dx} \qquad v^{IV} = \frac{P}{EJ} \rightarrow EJ\,v''' = -T$$

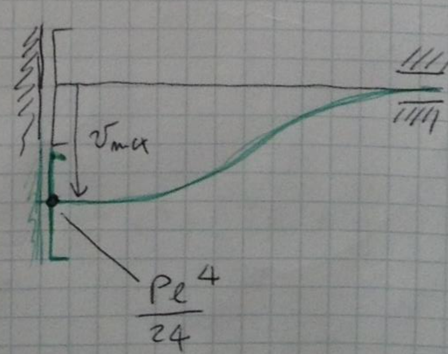
$$EJ\,v'''\Big|_{x=0} = -Px - A = 0 \rightarrow A = 0$$

$$v'(x=\ell) = 0 \longrightarrow \frac{P}{6}\ell^{\cancel{3}\,2} + B\cancel{\ell} = 0 \qquad B = -\frac{P\ell^2}{6}$$

$$v(x=\ell) = 0$$

$$\frac{P}{24}\ell^4 - \frac{P\ell^4}{12} + D = 0 \qquad D = \frac{P\ell^4}{24}$$

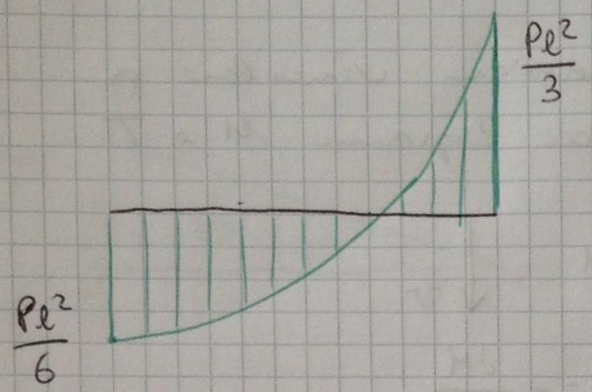
$$EJ\,v = \frac{P}{24}x^4 - \frac{P\ell^2}{12}x^2 + \frac{P\ell^4}{24}$$



$$v'' = -\frac{M}{EJ}$$

$$T = -EJ v''' = -Px$$

$$M = -EJv'' = -\frac{P}{2}x^2 + \frac{Pl^2}{6}$$



M_f