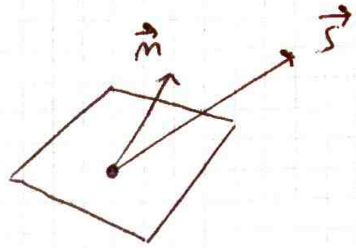
Perché trovare gli sforzi principali e le direzioni principali equivale a risolvere un problema agli autovalori?

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}$$



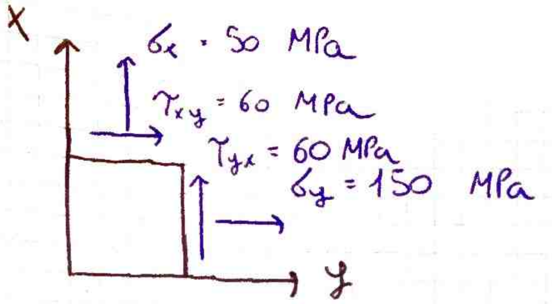
$$\overline{\overline{\sigma}}\,\vec{n} = \vec{S}$$

Se $\vec{n}$ definisce un piano principale, allora $\vec{S} \parallel \vec{n}_p$ su questo piano principale ($\vec{n}_p$), ovvero:

$$\begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} = \sigma_p \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}_p$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}_p = \sigma_p \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}_p$$

problema agli autovalori/autovettori



- Definire il tensore $\bar{\bar{\sigma}}$
- Calcolo sforzi principali
- Calcolo analitico delle direzioni principali
- Circonferenza del Mohr
- Verifica direzioni principali con circonferenza del Mohr

- Definizione tensore $\bar{\bar{\sigma}}$

Considero positivi σ $\leftarrow\square\rightarrow$ quando sono di trazione

Considero positivi τ se la normale del piano i (τ_{ij}) è concorde con il verso positivo di i e allo stesso tempo la direzione j dello sforzo è concorde con il verso positivo dell'asse

$$\bar{\bar{\sigma}} = \begin{bmatrix} 50 & 60 & 0 \\ 60 & 150 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Calcolo sforzi principali

Gli sforzi principali li possiamo determinare calcolando gli autovalori di $\bar{\bar{\sigma}}$

$$\det\left(\bar{\bar{\sigma}} - \sigma_p \bar{\bar{I}}\right) = 0 \qquad \det\left(\begin{bmatrix} 50 & 60 & 0 \\ 60 & 150 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \sigma_p & 0 & 0 \\ 0 & \sigma_p & 0 \\ 0 & 0 & \sigma_p \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} (50-\sigma_p) & 60 & 0 \\ 60 & (150-\sigma_p) & 0 \\ 0 & 0 & -\sigma_p \end{vmatrix} = 0 \qquad \left|\begin{array}{ccc|cc} (50-\sigma_p) & 60 & 0 & (50-\sigma_p) & 60 \\ 60 & (150-\sigma_p) & 0 & 60 & (150-\sigma_p) \\ 0 & 0 & -\sigma_p & 0 & 0 \end{array}\right|$$

$$\left[-\sigma_p(50-\sigma_p)(150-\sigma_p)\right] - \left[-3600\,\sigma_p\right] = 0$$

$$\sigma_p\left(3600 - 7500 + 50\,\sigma_p + 150\,\sigma_p - \sigma_p^2\right) = 0$$

$$\sigma_p\left(-\sigma_p^2 + 200\,\sigma_p - 3900\right) = 0 \qquad \sigma_{p,1} = 0$$

$$\sigma_{p,2-3} = \frac{-200 \pm \sqrt{40000 - 15600}}{-2} = \frac{-200 \pm 156,2}{-2} \begin{cases} \sigma_{p,2} = 21,9 \\ \sigma_{p,3} = 178,1 \end{cases}$$

$\sigma_I > \sigma_{II} > \sigma_{III} \rightarrow \sigma_I = 178{,}1 \text{ MPa}$

$\sigma_{II} = 21{,}9 \text{ MPa}$

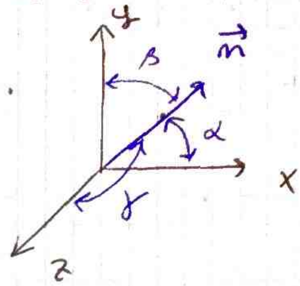
$\sigma_{III} = 0 \text{ MPa}$

- Calcolo analitico delle direzioni principali n_I, n_{II}, n_{III}

Autovettori di $\bar{\bar{\sigma}}$:

$$\begin{cases} (\bar{\bar{\sigma}} - \sigma_I \bar{\bar{I}})\vec{n}_I = 0 \\ (\bar{\bar{\sigma}} - \sigma_{II} \bar{\bar{I}})\vec{n}_{II} = 0 \\ (\bar{\bar{\sigma}} - \sigma_{III} \bar{\bar{I}})\vec{n}_{III} = 0 \end{cases}$$

$$n = \begin{bmatrix} i \\ l \\ m \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$$



α: Angolo tra direzione principale e asse x

β: " " " " " " y

γ: " " " " " " z

$\sigma_{III} = 0 \text{ MPa}$

$$\begin{bmatrix} 50 & 60 & 0 \\ 60 & 150 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ l \\ m \end{bmatrix} = \bar{0}$$

$50i + 60l = 0$

$60i + 150l = 0$

$\forall \gamma$ soddisfa le equazioni

$i = -\frac{60}{50} l \qquad -\frac{60^2}{50} l + 150 l = 0 \qquad \left(-\frac{60^2}{50} + 150\right) l = 0$

$l = 0 \rightarrow \cos\beta = 0 \qquad \beta = 90°$

$\cos\alpha = 0 \rightarrow \alpha = 90° \qquad i^2 + l^2 + m^2 = 1 \rightarrow m = 1 \rightarrow \gamma = 0$

$\sigma_{II} = 21{,}9 \text{ MPa}$

$$\begin{bmatrix} 50-21{,}9 & 60 & 0 \\ 60 & 150-21{,}9 & 0 \\ 0 & 0 & -21{,}9 \end{bmatrix} \begin{bmatrix} i \\ l \\ m \end{bmatrix} = \bar{0}$$

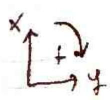
$28i + 60l = 0$

$60i + 128{,}1 l = 0$

linearmente dipendenti

$-21{,}9\, m = 0 \rightarrow \gamma = 90°$

$i^2 + l^2 + \underset{0}{\cancel{m^2}} = 1$



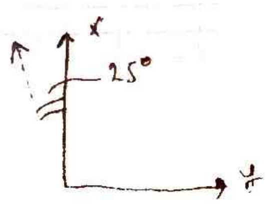
$i = -\frac{60}{28} l$

$\left(\frac{60^2}{28^2} + 1\right) l^2 = 1 \qquad l = \pm\sqrt{\frac{1}{\frac{60^2}{28^2}+1}} = \pm 0{,}424 \rightarrow \cos\beta = \pm 0{,}424$

$i = \mp \frac{60}{28} l = \mp 0{,}905$

$\cos\alpha = \mp 0{,}905$

$$\begin{cases} \begin{cases} \cos\alpha = -0{,}905 \\ \cos\beta = +0{,}424 \end{cases} \\ \begin{cases} \cos\alpha = +0{,}905 \\ \cos\beta = -0{,}424 \end{cases} \end{cases} \qquad \begin{cases} \alpha = \pm 155° \\ \beta = \pm 65° \end{cases} \quad \begin{cases} \alpha = \pm 25° \\ \beta = \pm 115° \end{cases}$$



$\sigma_I = 178{,}1$ MPa $\qquad (\bar{\bar{\sigma}} - \sigma_I \bar{\bar{I}})\,\bar{n}_I = 0$

$$\begin{bmatrix} 50-178{,}1 & 60 & 0 \\ 60 & 150-178{,}1 & 0 \\ 0 & 0 & -178{,}1 \end{bmatrix} \begin{bmatrix} i \\ l \\ m \end{bmatrix} = 0$$

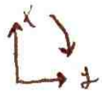
$-128{,}1\, i + 60\, l = 0$

$60\, i - 28{,}1\, l = 0$

$-178{,}1\, m = 0 \rightarrow \cos\gamma = 0 \rightarrow \gamma = 90°$

$i^2 + l^2 + \underbrace{\cancel{m^2}}_{=0} = 1$

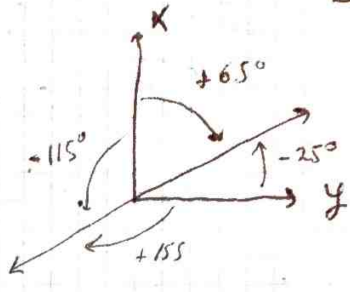
$i = \frac{28{,}1}{60}\, l = 0{,}47\, l$



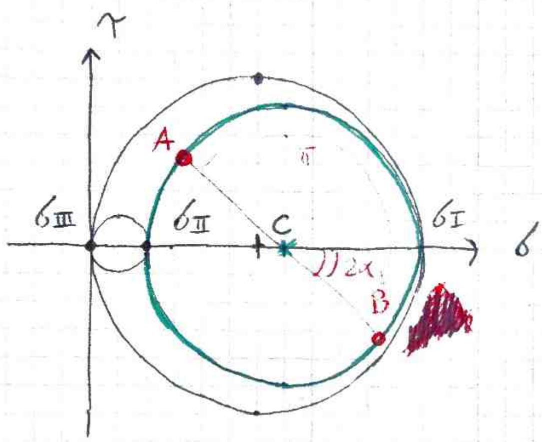
$0{,}47^2\, l^2 + l^2 = 1 \qquad l^2(1+0{,}47^2) = 1$

$$\begin{cases} l = \pm\sqrt{\dfrac{1}{1+0{,}47^2}} = \pm 0{,}9056 \\ i = \pm 0{,}4241 \end{cases}$$

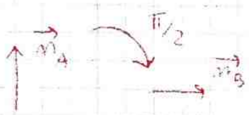
$$\begin{cases} \cos\beta = 0{,}9056 & \beta = \pm 25° \\ \cos\alpha = 0{,}4241 & \alpha = \pm 65° \\ \cos\beta = -0{,}9056 & \beta = \pm 155° \\ \cos\alpha = -0{,}4241 & \alpha = \pm 115° \end{cases}$$



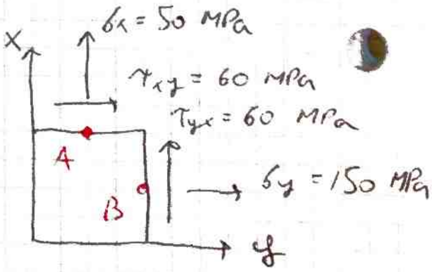
- Circonferenza del Mohr



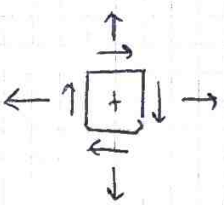
→ Conosco a priori che essendo lo stato di sforzo proposto essere piano → Asse z asse principale



→ Analizzo cerchio definito da σ_I e σ_{II}

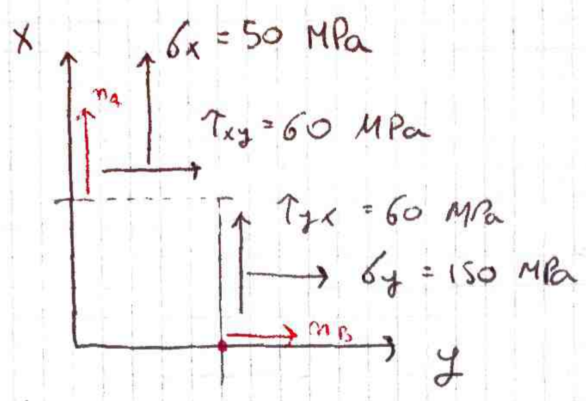


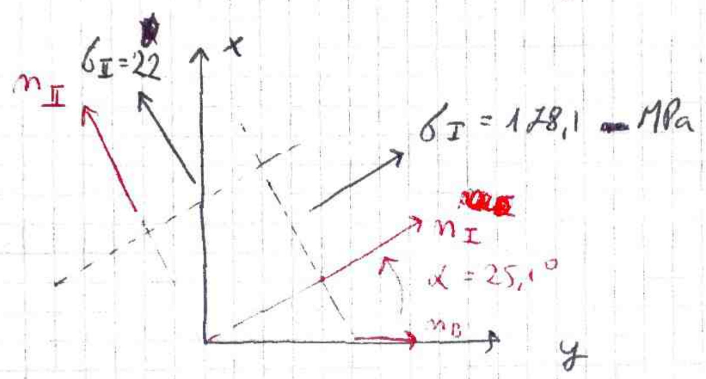
→ Convenzione:



$C = \dfrac{\sigma_I + \sigma_{II}}{2} = 100$ MPa $\qquad \overline{BC} = \sigma_I - C = 78{,}1$ MPa

$\overline{BC}\ \text{sen}\, 2\alpha = 60$ MPa $\rightarrow\ 2\alpha = 50{,}2°$





Sollecitazioni tangenziali massime:

$$\begin{cases} \tau_I = \dfrac{\sigma_I - \sigma_{III}}{2} = 89 \text{ MPa} \\ \tau_{II} = \dfrac{\sigma_I - \sigma_{II}}{2} = 78 \text{ MPa} \\ \tau_{III} = \dfrac{\sigma_{II} - \sigma_{III}}{2} = 11 \text{ MPa} \end{cases}$$

Sforzo tangenziale ottaedrale τ_{ott}:

$$\tau_{ott} = \frac{1}{3}\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}$$

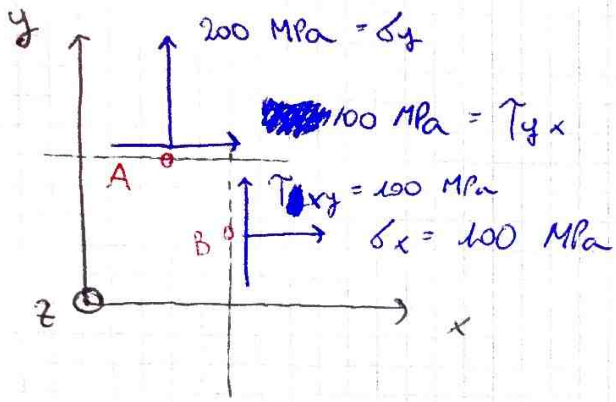
Dati:

$$\sigma_x = 100 \text{ MPa}, \quad \sigma_y = 200 \text{ MPa}, \quad \tau_{xy} = 100 \text{ MPa}$$
$$\sigma_z = \tau_{yz} = \tau_{zx} = 0 \text{ MPa}$$

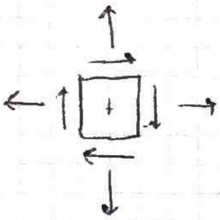
$E = 200'000$ MPa, $\nu = 0,3$ (acciaio)

Calcolare:

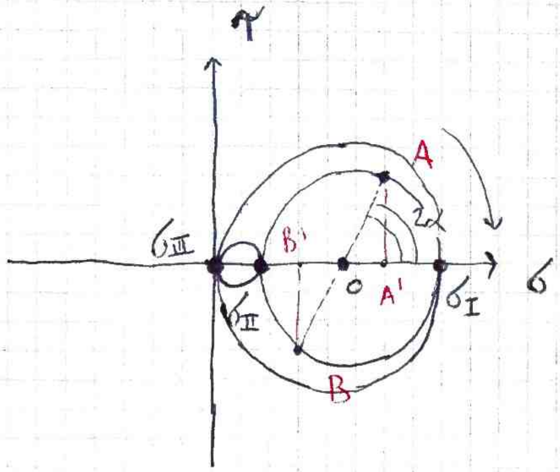
- Sollecitazioni principali (circonferenze del Mohr)
- Calcolare le componenti del tensore di deformazione
- Dimostrare che in uno stato di sforzo piano le deformazioni non sono piane.



La direzione // all'asse z (uscente) è una delle 3 direzioni principali con sforzo principale nullo.



Convenzione per tracciare le circonferenze di Mohr



A e B stanno su una circonferenza del Mohr poiché sappiamo che i piani di appartenenza di punti A e B siano ottenibili come rotazione intorno all'altro asse principale

Circonferenza definita da A e B.

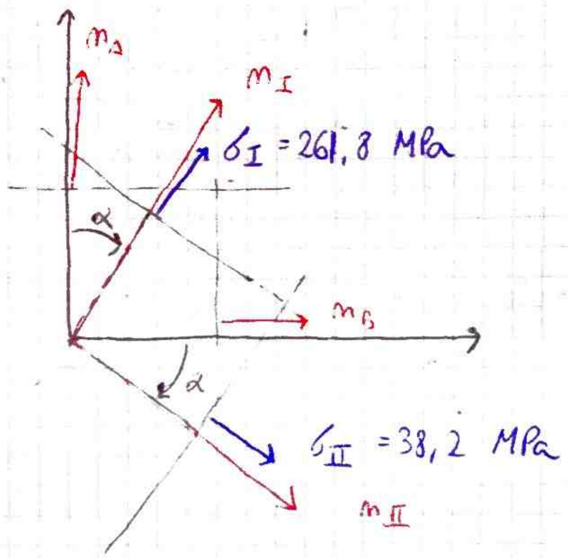
Raggio: $\sigma_0 = (\sigma_A + \sigma_B)/2 = (200 + 100)/2 = 150$ MPa

$$\overline{AO} = \sqrt{\overline{OA'}^2 + \overline{AA'}^2} = \sqrt{(\sigma_A - \sigma_0)^2 + (\tau_A)^2} = \sqrt{(200 - 150)^2 + (100)^2} =$$
$$= 111,8 \text{ MPa}$$

$$\sigma_I = \sigma_0 + \overline{OA} = 150 + 111,8 = 261,8 \text{ MPa}$$

$$\sigma_{II} = \sigma_0 - \overline{OA} = 150 - 111,8 = 38,2 \text{ MPa}$$

$$\tan 2\alpha = \frac{\overline{AA'}}{\overline{OA'}} = \frac{\sigma_A}{\sigma_A - \sigma_O} \Rightarrow 2\alpha = 66{,}2° \qquad \alpha = 33{,}1°$$



Per materiale isotropo → 2 costanti elastiche (E, ν)
La terza si ottiene come:

$$G = \frac{E}{2(1+\nu)} = 76923 \text{ MPa}$$

Legame elastico tra sforzi e deformazioni:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & & [0] & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/G & 0 & 0 \\ & [0] & & 0 & 1/G & 0 \\ & & & 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \begin{matrix} \\ \\ = 0 \\ \\ = 0 \\ = 0 \end{matrix}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}\sigma_y = 0{,}0002 \text{ m/m} = 0{,}02\ \%$$

$$\varepsilon_y = -\frac{\nu}{E}\sigma_x + \frac{\sigma_y}{E} = 0{,}00085 \text{ m/m} = 0{,}085\ \%$$

$$\varepsilon_z = -\frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y = -0{,}00045 \text{ m/m} = -0{,}045\ \%$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0{,}0013 \text{ mm/mm} = 0{,}13\ \%$$