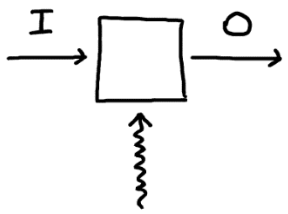
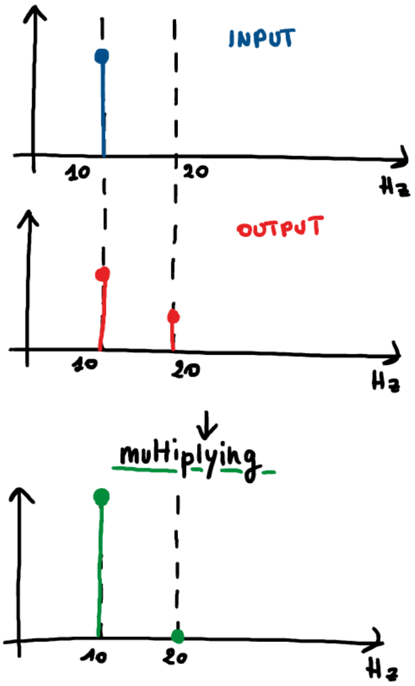
CROSS SPECTRUM AND COHERENCE

Introduction



In previous lessons we have studied and discussed about the FRF as the ratio between the output over the input. By doing this we have given as granted that we have only one input and, in particular, we have only the input in which we are interested. Actually, most of the time we have some spurious input coming from other phenomena, so, generally we have more than one single input. The bad thing is that most of the time we don't realise that we have these unwanted inputs.

Cross spectrum



In order to understand how we can insert this aspect into the model that we have described until now let's image to have two spectra: one representing the input and one representing the output. In the input spectrum we have only one component at $10\,Hz$. In the second spectrum, the one that represents the output, we have 2 components: one at $10\,Hz$ and one at $20\,Hz$.
Since in the input we have only two components we can easily conclude that the component of the output at $20\,Hz$ is due to the unwanted and spurious input (disturbances or non-linearities). Since we have in the output this unwanted component that is not present in the input, we can say that the model that we have discussed until now is not perfectly correct. In addition to this we cannot think of doing the FRF because for the component at $20\,Hz$ we would have an amplitude divided by a 0 (we don't have any component at $20\,Hz$ in the input spectrum) and this would lead us to have an infinite value; this is, of course, not true and so we have for sure measured something wrong or at least we have forgotten to measure something at the frequency that is missing. So, now our question is how we can trace the presence of the component that is missing in the input but is present into the output? We can think of solving this problem by multiplying the two spectra. With this sentence we mean multiply frequency by frequency each component of the two spectra. By doing this we end up with a plot in which we have only the component related to the input. The output of this multiplication is a complex vector and it's called cross spectrum. The cross spectrum is a function allowing to get information on the "existing relation" between two analysed channels. By a mathematical point of view, it can be computed as the product between one spectrum (in our case the output spectrum) and the complex conjugate of another spectrum (in our case the complex conjugate of the input spectrum):

$$S_{AB}(f) = B^*(f) \cdot A(f)$$

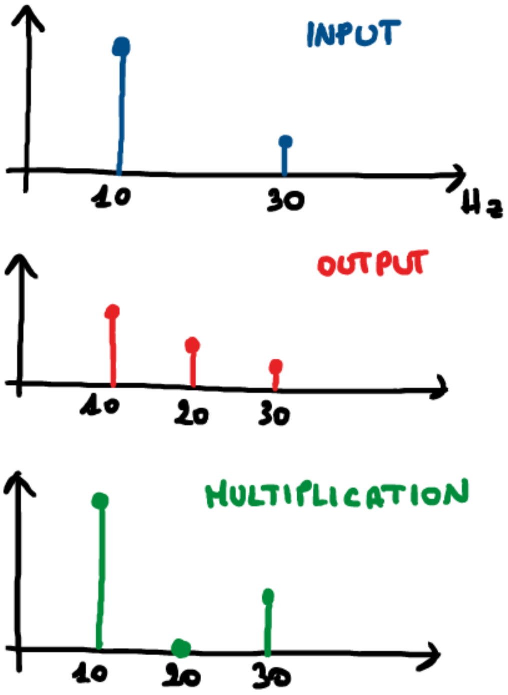
where $B^*(f)$ is the complex conjugate of the input spectrum and $A(f)$ is the output spectrum; the fact that we have the complex conjugate comes to the fact that it helps on determining the phase delaying between one quantity and the other, in measurement it's not very useful but it's very important in the electrical fields. This formula, in fact, comes from the electricity field: if we think of this field, we would apply the cross spectrum on the voltage and the current and we would end up with the spectrum of the electrical power.

Coherence

Let's now image to have a second component in the input at $30\,Hz$ which amplitude is quite small; also, in the output spectrum we have this component at $30\,Hz$. In this case if we multiply the two amplitude of the components, we'll not have zero in the cross spectrum since none of the two components, in input and output are zero. In this case we cannot conclude anything, we cannot even compare this component with the one at 10 Hz and so we cannot say if it best or worse than that component. We need a further step to make this technique more effective in order to better understand when a component in the cross spectrum is reliable or not. To solve this problem, we need to normalise the cross spectrum so that we can have a sort of alignment. By doing this we are also able to understand how good or how bad is the output that we obtain with respect to the input. To normalise we need to:

$$0 \leq \gamma_{AB}^{2}(f) = \frac{|S_{AB}(f)|^{2}}{S_{AA}(f) \cdot S_{BB}(f)} \leq 1$$

Where S_{AB} is the cross spectrum, obtained by using the formula previously provided, S_{AA} is the autospectrum of the input and S_{BB} is the autospectrum of the output. Let's remember that the autospectrum is a real value obtained as the product between the spectrum and the complex conjugate of the same spectrum:



$$S_{AA}(f) = A(f) \cdot A^{*}(f)$$

$$S_{BB}(f) = B(f) \cdot B^{*}(f)$$

We can define γ_{AB}^{2} as the coherence. This value is used to understand when we have coherence between the input and the output. In other words, it's a dimensionless value that is used to determine how two signals are linearly related. We can point out that the denominator of the formula represents the maximum energy that we can have. In addition to this we can also say that the autospectrum and the coherence are real values whereas the cross spectrum is a complex value; anyway, all these quantities are function of the frequency. It can be a value between 0 and 1: when we have 1 we have maximum energy and so we have the perfect correspondence between the input and the output, on the other hand when we have 0 we don't have any energy and this means having no correspondence between the input and the output. The coherence is always equal to 1 when computed on a single time record; let's try to understand why with a demonstration:

$$\gamma_{AB}^{2}(f) = \frac{|S_{AB}(f)|^{2}}{S_{AA}(f) \cdot S_{BB}(f)} =$$

$$= \frac{|S_{AB}(f)|^{2}}{[A(f) \cdot A^{*}(f)] \cdot [B(f) \cdot B^{*}(f)]} =$$

$$= \frac{|S_{AB}(f)|^{2}}{S_{AB}(f) \cdot S_{BA}(f)} =$$

$$= \frac{S_{AB}^{*}(f) \cdot S_{AB}(f)}{S_{AB}(f) \cdot S_{BA}(f)} =$$

$$= \frac{S_{AB}^{*}(f)}{S_{BA}(f)} =$$

$$= \frac{S_{BA}(f)}{S_{BA}(f)} = 1$$

For this reason, it must be always computed using the averaged cross spectra and auto spectra. Let's now try to understand why the cross spectrum helps us to understand if there is a linear relationship between input and output:

$$S_{AB} = A^*(f) \cdot B(f) =$$

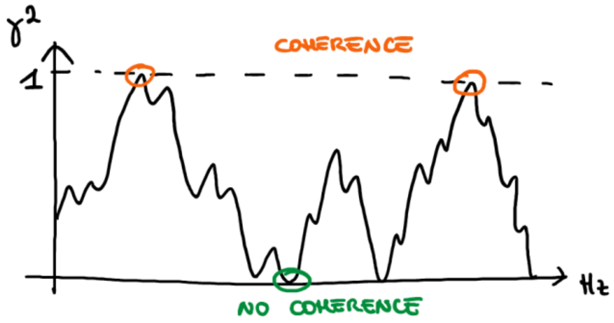
$$= \left[|A(f)|e^{i\varphi_A}\right]^* \cdot \left[|B(f)|e^{i\varphi_B}\right] =$$

$$= |A(f)|e^{-i\varphi_A} \cdot |B(f)|e^{i\varphi_B} =$$

$$= |B(f)||A(f)|\, e^{i\varphi_B - i\varphi_A} =$$

$$= |B(f)||A(f)|\, e^{i\varphi_{AB}}$$

We can now depict the coherence in a plot to better understand in which frequency we have coherence and in which we don't.



Example (something similar can be asked in the exam): let's imagine having a car with an engine and the exhaust[1]. Someone complains that the car is too noisy, so we need to work on it to reduce the question; the problem is that we don't have unlimited money but a fix quantity that allows us to act only on one of the two components. So, we can choose of working on the engine or on the exhaust, but not on both of them. In order to understand on which part, I'm supposed to work we must first discriminate and identify what of the two components is the most responsible one for the noise. To do that we must be able to understand what part of the total noise that I can acquire comes from the engine and what part come from the exhaust. So, we of course, the coherence will help me to make this discrimination. The problem is trying to understand the coherence of what? What are the signals on which I can then compute the coherence? What are the inputs?

Talking about the output we can put a microphone near the car and take the measurement. A more difficult choice would be the place in which acquire the input; at first, we can think of putting a microphone near the exhaust and consider it as the input or, vice versa, put a microphone near the engine. The problem of both these acquisitions is the fact that both of them will be influenced by the other one. So, in this case we will have always a pollution of what comes from the other component. To solve this problem, we can think of not using a microphone but using an accelerometer to measure the vibration. We can place the accelerometer in the engine so that the vibration will less influenced by the exhaust. We can then take the coherence between the accelerometer and the microphone. We can fix a threshold and say that if the coherence is bigger than this threshold the vibration (and so the noise) would come from the the engine, otherwise form the exhaust.

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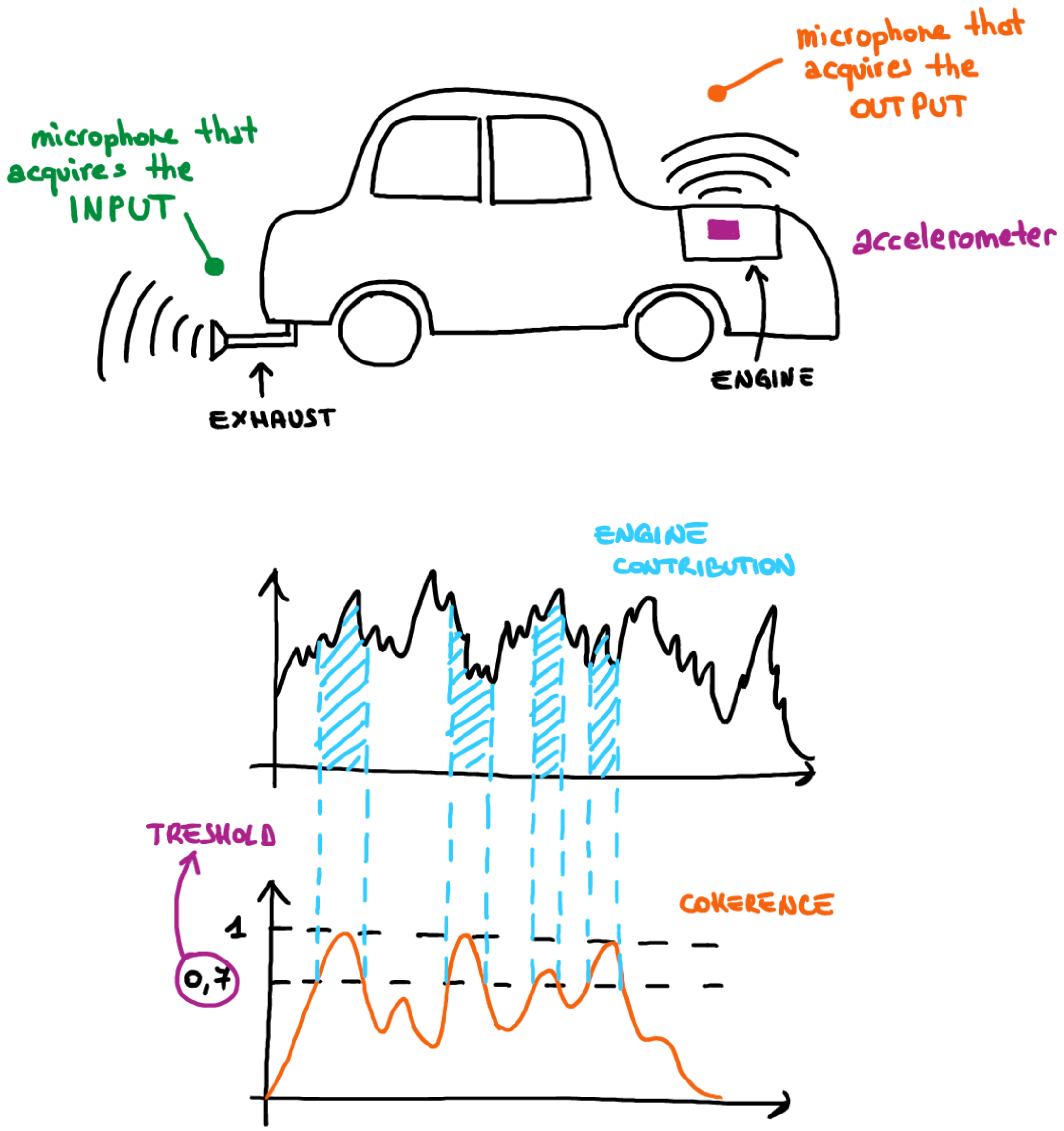
[1] Tubo di scarico/scappamento


microphone that acquires the INPUT
microphone that acquires the OUTPUT
accelerometer
EXHAUST
ENGINE
ENGINE CONTRIBUTION
TRESHOLD
1
0,7
COHERENCE