Exercise 1.5: After lunch one day, Alice suggests to Bob the following method to determine who pays. Alice pulls three six-sided dice from her pocket. These dice are not the standard dice, but have the following numbers on their faces:

- die A – 1, 1, 6, 6, 8, 8;
- die B – 2, 2, 4, 4, 9, 9;
- die C – 3, 3, 5, 5, 7, 7.

The dice are fair, so each side comes up with equal probability. Alice explains that Alice and Bob will each pick up one of the dice. They will each roll their die, and the one who rolls the lowest number loses and will buy lunch. So as to take no advantage, Alice offers Bob the first choice of the dice.

(a) Suppose that Bob chooses die A and Alice chooses die B. Write out all of the possible events and their probabilities, and show that the probability that Alice wins is greater than 1/2.

(b) Suppose that Bob chooses die B and Alice chooses die C. Write out all of the possible events and their probabilities, and show that the probability that Alice wins is greater than 1/2.

(c) Since die A and die B lead to situations in Alice's favor, it would seem that Bob should choose die C. Suppose that Bob does choose die C and Alice chooses die A. Write out all of the possible events and their probabilities, and show that the probability that Alice wins is still greater than 1/2.

$X = \{\text{ESITO LANCIO DADO A}\} = \{1, 6, 8\}$ $\quad \mathbb{P}[X = x] = \frac{1}{3}$

$Y = \{\text{ESITO LANCIO DADO B}\} = \{2, 4, 9\}$ $\quad \mathbb{P}[Y = y] = \frac{1}{3}$

$Z = \{\text{ESITO LANCIO DADO C}\} = \{3, 5, 7\}$ $\quad \mathbb{P}[Z = z] = \frac{1}{3}$

$A = \{\text{VINCE ALICE}\}$; $B = \{\text{VINCE BOB}\}$

X, Y, Z INDIPENDENTI $\Rightarrow$

$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y] = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

IDEM PER LE COPPIE X, Z E Y, Z.

CASO (a) LA MATRICE DELLE PROB. CONGIUNTE $\mathbb{P}[X=x, Y=y]$ È :

Y (ALICE) \ X (BOB)	1	6	8
2	1/9	1/9	1/9
4	1/9	1/9	1/9
9	1/9	1/9	1/9

POICHÉ $A = \{X < Y\}$ E

$\mathbb{P}[A] = \mathbb{P}[X < Y] = \sum_{x<y} \frac{1}{9}$ VEDIAMO QUANDO $X<Y$:

Y (ALICE) \ X (BOB)	1	6	8
2	A	B	B
4	A	B	B
9	A	A	A

ALICE VINCE IN 5 CASI (SU 9) QUINDI

$$\mathbb{P}[A] = 5 \cdot \frac{1}{9} = \frac{5}{9}$$

(b) E (c) SONO ANALOGHI.

45. A product is classified according to the number of defects it contains and the factory that produces it. Let X_1 and X_2 be the random variables that represent the number of defects per unit (taking on possible values of 0, 1, 2, or 3) and the factory number (taking on possible values 1 or 2), respectively. The entries in the table represent the joint possibility mass function of a randomly chosen product.

$X_1 \backslash X_2$	1	2
0	$\frac{1}{8}$	$\frac{1}{16}$
1	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{3}{16}$	$\frac{1}{8}$
3	$\frac{1}{8}$	$\frac{1}{4}$

(a) Find the marginal probability distributions of X_1 and X_2.
(b) Find $E[(X_1)]$, $E[(X_2)]$, $\text{Var}(X_1)$, $\text{Var}(X_2)$, and $\text{Cov}(X_1, X_2)$.

$$E[X] = \sum_i x_i P\{X = x_i\}$$

$$\text{Var}(X) = E[(X-\mu)^2]$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

$$\text{Cov}(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$$

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$\text{Cov}(X,Y) = \sum_{i,j} p_{X,Y}(x_i, y_j) x_i y_j - \sum_i p_X(x_i) x_i * \sum_j p_Y(y_j) y_j$$

TRASCRIVIAMO LA MATRICE DELLE PROB. CONGIUNTE E AGGIUNGIAMO LE PROB. MARGINALI (SOMME PER RIGHE E PER COLONNE). AD ES. $\mathbb{P}[X_1=0] = 2/16 + 1/16 = 3/16$

$\mathbb{P}[X_1,X_2]$	1	2	TOT
0	2/16	1/16	3/16
1	1/16	1/16	2/16
2	3/16	2/16	5/16
3	2/16	4/16	6/16
TOT	8/16	8/16	

POI USIAMO LA MATRICE PER CALCOLARE GLI INDICATORI RICHIESTI:

$$\mathbb{E}[X_1] = 0 \cdot \frac{3}{16} + 1 \cdot \frac{2}{16} + 2 \cdot \frac{5}{16} + 3 \cdot \frac{6}{16} = \frac{2}{16} + \frac{10}{16} + \frac{18}{16} = \frac{30}{16} = \frac{15}{8}$$

$$\mathbb{E}[X_2] = \frac{8}{16} + \frac{16}{16} = \frac{24}{16} = \frac{3}{2}$$

$$\mathbb{E}[X_1X_2] = \frac{0\cdot1\cdot2 + 0\cdot2\cdot1 + 1\cdot1\cdot1 + 1\cdot2\cdot1 + 2\cdot1\cdot3 + 2\cdot2\cdot2 + 3\cdot1\cdot2 + 3\cdot2\cdot4}{16}$$

$$= \frac{1+2+6+8+6+24}{16} = \frac{47}{16}$$

PER LA VARIANZA CI SERVONO $\mathbb{E}X_1^2$ E $\mathbb{E}X_2^2$. RISCRIVIAMO PER PIGNOLERIA LA MATRICE CON X_i^2 AL POSTO DELLE X_i:

$\mathbb{P}[X_1^2, X_2^2]$	1^2	2^2	TOT
0^2	$2/16$	$1/16$	$3/16$
1^2	$1/16$	$1/16$	$2/16$
2^2	$3/16$	$2/16$	$5/16$
3^2	$2/16$	$4/16$	$6/16$
TOT	$8/16$	$8/16$	

$$\mathbb{E}[X_1^2] = 1^2 \cdot \frac{2}{16} + 2^2 \cdot \frac{5}{16} + 3^2 \cdot \frac{6}{16} = \frac{2 + 20 + 54}{16} = \frac{76}{16}$$

$$\mathbb{E}[X_2^2] = 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2} = \frac{5}{2}$$

INFINE CALCOLIAMO VAR E COV USANDO I RISULTATI INTERMEDI APPENA RICAVATI:

$$\mathrm{Var}[X_1] = \mathbb{E}[X_1^2] - (\mathbb{E}[X])^2$$

$$= \frac{76}{16} - \left(\frac{15}{8}\right)^2 = \frac{304}{64} - \frac{225}{64} = \frac{79}{64}$$

$$\mathrm{Var}[X_2] = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{10}{4} - \frac{9}{4} = \frac{1}{4}$$

$$\mathrm{Cov}[X_1, X_2] = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$$

$$= \frac{47}{16} - \frac{15}{8} \cdot \frac{3}{2} = \frac{1}{8}$$

Si consideri la famiglia di funzioni reali di variabile reale ad un parametro $c \in \mathbb{R}$:

$$f_c(x) := \begin{cases} e^{-cx} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

1. Per quali valori di c la funzione rappresenta la densità di una variabile assolutamente continua?
2. Al variare dei valori di c ammissibili calcolati in precedenza e denotato con X la variabile, se ne calcoli il valore atteso e la funzione di ripartizione
3. Quanto vale $\mathbb{P}[X \geq t+s \mid X \geq t]$ con $t, s \geq 0$?
4. Sia $\{X_i\}_{i=1}^{100}$ una successione di variabili i.i.d. con densità $f_1(x)$. Assumendo le uguaglianze $\mathbb{E}X_i = VarX_i = 1$ calcolare approssimativamente $\mathbb{P}\left[\sum_{i=1}^{100} X_i \geq 100\right]$.

① DEVE ESSERE $f_c(x) \geq 0 \quad \forall x \in \mathbb{R}$. VERO $\forall c \in \mathbb{R}$.

INOLTRE $\int_{\mathbb{R}} f_c(x)\,dx = 1$ OVVERO

$$1 = \int_{-\infty}^{+\infty} f_c(x)\,dx = \int_0^{+\infty} e^{-cx}\,dx = -\frac{1}{c} e^{-cx}\Big|_0^{+\infty} = \frac{1}{c}$$ DA CUI $c = 1$

NOTA: L'INTEGRALE CONVERGE SOLO SE $c > 0 \Rightarrow c = 1$ È COMPATIBILE.

② $$\mathbb{E}X = \int_{\mathbb{R}} x f_1(x)\,dx = \int_0^{+\infty} x e^{-x}\,dx$$ $\Rightarrow$ PER PARTI

$f = x \qquad g' = e^{-x}$

$f' = 1 \qquad g = -e^{-x}$

$$\Rightarrow \mathbb{E}X = -xe^{-x}\Big|_0^{+\infty} - \int_0^{+\infty} -e^{-x}\,dx = -e^{-x}\Big|_0^{\infty} = 1$$

INOLTRE $$F_X(x) = \begin{cases} 0 & \text{PER } x < 0 \\ \int_0^x f_1(t)\,dt & \text{PER } x \geq 0 \end{cases}$$

OVVERO $$F_X(x) = \begin{cases} 0 & \text{PER } x < 0 \\ \int_0^x e^{-t}\,dt = -e^{-t}\Big|_0^x = 1 - e^{-x} & \text{PER } x \geq 0 \end{cases}$$

IN FORMA COMPATTA: $F_X(x) = \mathbb{1}_{(0,+\infty)}(x) \cdot (1 - e^{-x})$

(3) $$\mathbb{P}[X \geq t+s \mid X \geq t] \underset{\substack{\text{PER DEF.}\\ \text{DI PROB.}\\ \text{CONDIZIONATA}}}{=} \frac{\mathbb{P}[X \geq t+s \cap X \geq t]}{\mathbb{P}[X \geq t]} =$$

$$= \frac{\mathbb{P}[X \geq t+s]}{\mathbb{P}[X \geq t]} = \frac{1 - F_X(t+s)}{1 - F_X(t)} = \frac{e^{-t-s}}{e^{-t}} = e^{-s} =$$

NOTA: SIGNIFICA ASSENZA DI MEMORIA

$$= \mathbb{P}[X \geq s]$$

(4) EVIDENTEMENTE È RICHIESTA UN'APPROSSIMAZIONE VIA TCL.
POICHÈ $n = 100$ LA CONDIZIONE È SODDISFATTA

$$n \geq 30 \quad H_n = X_1 + \cdots + X_n \approx Z \sim \mathcal{N}(n\mu, n\sigma^2); \quad \mathbb{P}(H_n \leq t) \approx \Phi\left(\frac{t - n\mu}{\sqrt{n}\sigma}\right)$$

QUINDI $$\mathbb{P}[H_n \geq 100] \approx 1 - \Phi\left(\frac{100 - 100 \cdot 1}{\sqrt{100 \cdot 1}}\right) = 1 - \Phi(0) = \frac{1}{2}$$